

The Drazin spectrum of tensor product of Banach algebra elements and elementary operators

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Abstract

Given unital Banach algebras A and B and elements $a \in A$ and $b \in B$, the Drazin spectrum of $a \otimes b \in A \overline{\otimes} B$ will be fully characterized, where $A \overline{\otimes} B$ is a Banach algebra that is the completion of $A \otimes B$ with respect to a uniform crossnorm. To this end, however, first the isolated points of the spectrum of $a \otimes b \in A \overline{\otimes} B$ need to be characterized. On the other hand, given Banach spaces X and Y and Banach space operators $S \in L(X)$ and $T \in L(Y)$, using similar arguments the Drazin spectrum of $\tau_{ST} \in L(L(Y, X))$, the elementary operator defined by S and T , will be fully characterized.

Keywords: Drazin spectrum; Banach algebra; tensor product; Banach space; elementary operator.

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1. Introduction

The relationships among tensor products and spectral theory have been extensively studied. For example, it is well known that given unital Banach algebras A and B , if $A \overline{\otimes} B$ is a Banach algebra that is the completion of the algebraic tensor product $A \otimes B$ with respect to a uniform crossnorm, then $\sigma(a \otimes b) = \sigma(a)\sigma(b)$, where $a \in A$, $b \in B$ and $\sigma(\cdot)$ is the usual spectrum. When X and Y are Banach spaces, $X \overline{\otimes} Y$ is a completion of the algebraic tensor product $X \otimes Y$ with respect to an appropriate norm and $S: X \rightarrow X$ and $T: Y \rightarrow Y$ are Banach space operators, then $\sigma_*(S \otimes T) = \sigma_*(S)\sigma_*(T)$, where $\sigma_*(\cdot)$ is either the approximate point spectrum or the defect spectrum ([15]). However, if $\sigma_+(\cdot)$ denotes either the Fredholm or the Browder spectrum, then the following formula holds: $\sigma_+(S \otimes T) = \sigma(S)\sigma_+(T) \cup \sigma_+(S)\sigma(T)$ ([11, 6]). Formulas similar to the last one were proved also for other spectra such as the upper and lower Fredholm spectra and the Browder approximate point spectrum, if instead of the usual spectrum, the approximate point spectrum is considered ([15, 9]). On the other hand, it is well known that the spectral theories of tensor products of operators and elementary operators are deeply connected ([5, 11]). Now well, similar formulas for the aforementioned spectra of elementary operators were proved in [11, 4].

The first objective of this article is to fully characterize $\sigma_{DR}(a \otimes b)$, the Drazin spectrum of $a \otimes b \in A \overline{\otimes} B$, in terms of the spectrum and the Drazin spectrum of $a \in A$ and $b \in B$, where A , B and $A \overline{\otimes} B$ are as at the beginning of the previous paragraph. In fact, in section 4 the relationship between $\sigma_{DR}(a \otimes b)$ and $\mathbb{D} = \sigma(a)\sigma_{DR}(b) \cup \sigma_{DR}(a)\sigma(b)$ will be studied. In particular, it will prove that $\sigma_{DR}(a \otimes b) \subseteq \mathbb{D}$ and $\sigma_{DR}(a \otimes b) \setminus \{0\} = \mathbb{D} \setminus \{0\}$, necessary and sufficient conditions to characterize when $\sigma_{DR}(a \otimes b)$ and \mathbb{D} coincide will be given and clearly, when $\sigma_{DR}(a \otimes b) \subsetneq \mathbb{D}$, then $\mathbb{D} = \sigma_{DR}(a \otimes b) \cup \{0\}$, $0 \notin \sigma_{DR}(a \otimes b)$. On the other hand, the second objective of this

work is to study the Drazin spectrum of elementary operators between Banach spaces, which will be fully characterized in section 5 using arguments similar to the ones in section 4.

However, to this end, the isolated points of $\sigma(a \otimes b)$, $a \otimes b \in A \overline{\otimes} B$, A , B and $A \overline{\otimes} B$ as before, need to be characterized in terms of the poles and the complements of the poles in the isolated points of the spectrum of both $a \in A$ and $b \in B$. This will be done in section 3, after having recalled in section 2 the preliminary definitions and results that will be used in the present work. In addition, this characterization will deepen the knowledge of the isolated points of the spectrum of elementary tensors obtained in [13, 10].

It is worth noticing that the Drazin spectrum is a key notion in the research area of Weyl's and Browder's theorems and their generalizations. What is more, in the recent past (generalized) Weyl's and (generalized) Browder's theorems of both tensor product operators and elementary operators have been studied ([23, 1, 19, 8, 9, 4, 13, 10]). On the other hand, the set of isolated points of the spectrum is another central notion in this area of research. Therefore, the results obtained in the present work, apart from their interest related to spectral theory and the Drazin inverse, also present an interest related to the area of Weyl's and Browder's theorems.

2. Preliminary definitions and results

From now on A will denote a unital Banach algebra with unit e . Recall that the element $a \in A$ is said to be *Drazin invertible*, if there exists a necessarily unique $b \in A$ and some $m \in \mathbb{N}$ such that

$$a^m b a = a^m, \quad b a b = b, \quad a b = b a.$$

The least non-negative integer m for which the above equations holds is called the *index* of a . In addition, if the Drazin inverse of a exists, then it will be denoted by a^D ; concerning this research area see for example [7, 16, 22, 2, 3].

On the other hand, the notion of *regularity* was introduced and studied in [18, 20]. Recall that given a unital Banach algebra A and a regularity $\mathcal{R} \subseteq A$, the *spectrum derived from the regularity* \mathcal{R} is defined as $\sigma_{\mathcal{R}}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin \mathcal{R}\}$, where $a \in A$ and $a - \lambda$ stands for $a - \lambda e$, see [18]. In addition, the *resolvent set of a defined by the regularity \mathcal{R}* is the set $\rho_{\mathcal{R}}(a) = \{\lambda \in \mathbb{C} : a - \lambda \in \mathcal{R}\}$. Naturally, when $\mathcal{R} = A^{-1}$, the set of all invertible elements of A , $\sigma_{A^{-1}}(a) = \sigma(a)$, the spectrum of a , and $\rho_{A^{-1}}(a) = \rho(a) = \mathbb{C} \setminus \sigma(a)$, the resolvent set of a . Next consider the set $\mathcal{DR}(A) = \{a \in A : a \text{ is Drazin invertible}\}$. According to [2, Theorem 2.3], $\mathcal{DR}(A)$ is a regularity. This fact led to the following definition, see [2].

Definition 2.1. *Let A be a unital Banach algebra. The Drazin spectrum of $a \in A$ is the set*

$$\sigma_{\mathcal{DR}}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin \mathcal{DR}(A)\}.$$

Naturally $\sigma_{\mathcal{DR}}(a) \subseteq \sigma(a)$, $\rho(a) \subseteq \rho_{\mathcal{DR}}(a)$ and according to [18, Theorem 1.4], the Drazin spectrum of a Banach algebra element satisfies the spectral mapping theorem for analytic functions defined on a neighbourhood of the usual spectrum which are non-constant on each component of its domain of definition (see also [2, Corollary 2.4]). In addition, according to [2, Proposition 2.5], $\sigma_{\mathcal{DR}}(a)$ is a closed subset of \mathbb{C} . Next a description of the spectrum and the Drazin spectrum in terms of the isolated and limit points of the spectrum given in [3] will be recalled. However, to this end, some preparation is needed. First of all, if $K \subseteq \mathbb{C}$, then, $\text{iso } K$ will stand for the isolated points of K and $\text{acc } K = K \setminus \text{iso } K$.

Let A be a unital Banach algebra and $a \in A$. As in the case of the spectrum of a Banach space operator, the resolvent function of a , $R(\cdot, a) : \rho(a) \rightarrow A$, is holomorphic and $\text{iso } \sigma(a)$ coincides with the set of isolated singularities of $R(\cdot, a)$. Furthermore, as in the case of an operator, see [24, p. 305], if $\lambda_0 \in \text{iso } \sigma(a)$, then it is possible to consider the Laurent expansion of $R(\cdot, a)$ in

terms of $(\lambda - \lambda_0)$. In fact,

$$R(\lambda, a) = \sum_{n \geq 0} a_n (\lambda - \lambda_0)^n + \sum_{n \geq 1} b_n (\lambda - \lambda_0)^{-n},$$

where a_n and b_n belong to A and are obtained in an standard way using the functional calculus. In addition, this representation is valid when $0 < |\lambda - \lambda_0| < \delta$, for any δ such that $\sigma(a) \setminus \{\lambda_0\}$ lies outside the circle $|\lambda - \lambda_0| = \delta$. What is more important, the discussion of [24, pp. 305 and 306] can be repeated for elements in a unital Banach algebra. Consequently, λ_0 will be called a *pole of order p of $R(\cdot, a)$* , if there is $p \geq 1$ such that $b_p \neq 0$ and $b_m = 0$, for all $m \geq p + 1$. The set of poles of a will be denoted by $\Pi(a)$; clearly $\Pi(a) \subseteq \text{iso } \sigma(a)$.

Let X be a Banach space and denote by $L(X)$ the algebra of bounded and linear maps defined on and with values in X . If $T \in L(X)$, then $N(T)$ and $R(T)$ will stand for the null space and the range of T respectively. Recall that the *descent* and the *ascent* of $T \in L(X)$ are $d(T) = \inf\{n \geq 0: R(T^n) = R(T^{n+1})\}$ and $a(T) = \inf\{n \geq 0: N(T^n) = N(T^{n+1})\}$ respectively, where if some of the above sets is empty, its infimum is then defined as ∞ , see for example [20, 2]. Now well, according to [16, Theorem 4], $\Pi(T) = \{\lambda \in \sigma(a): a(T - \lambda) \text{ and } d(T - \lambda) \text{ are finite}\}$. What is more, if A is a unital Banach algebra and $a \in A$, then, according to [3, Theorem 11], $\Pi(a) = \Pi(L_a) = \Pi(R_a)$, where $L_a, R_a \in L(A)$ are the operators defined by left and right multiplication, i.e., given $x \in A$, $L_a(x) = ax$ and $R_a(x) = xa$ respectively.

Now well, according to [3, Theorem 12],

$$\sigma(a) = \sigma_{\mathcal{DR}}(a) \cup \Pi(a), \quad \sigma_{\mathcal{DR}}(a) = \text{acc } \sigma(a) \cup I(a), \quad \Pi(a) = \sigma(a) \cap \rho_{\mathcal{DR}}(a),$$

where $I(a) = \text{iso } \sigma(a) \setminus \Pi(a)$. Note that $\sigma_{\mathcal{DR}}(a) \cap \Pi(a) = \emptyset$ and $(\text{acc } \sigma(a)) \cap I(a) = \emptyset$.

Finally, if A and B are unital Banach algebras, then $A \otimes B$ will stand for the algebraic tensor product of A and B , while $\overline{A \otimes B}$ will denote a Banach algebra that is the completion of $A \otimes B$ with respect to a uniform crossnorm; see for example [12, Chapter 11, Section 7].

3. Isolated points

To prove the key results of this article, the isolated points of the spectrum of the tensor product of two Banach algebra elements need to be characterized first. To this end, however, some preparation is needed.

Remark 3.1. Let A be a unital Banach algebra and consider $a \in A$. Recall that if $p^2 = p \in A$, then pAp is a unital Banach algebra with unit p .

(i) Necessary and sufficient for $\lambda \in \sigma(a)$ to belong to $\Pi(a)$ is that there exists $0 \neq p^2 = p \in A$ such that $ap = pa$, $(a - \lambda)p$ is nilpotent and $(a - \lambda + p) \in A^{-1}$. Note in particular that $\sigma(a) = \Pi(a) = \{\lambda\}$ if and only if $p = e$, which is equivalent to the fact that $a - \lambda$ is nilpotent.

(ii) The number $\lambda \in \sigma(a)$ belongs to $I(a)$ if and only if there is $0 \neq p^2 = p \in A$ such that $ap = pa$, $(a - \lambda)p$ is quasi-nilpotent but not nilpotent and $(a - \lambda + p) \in A^{-1}$. What is more, as in the previous statement, $\sigma(a) = I(a) = \{\lambda\}$ if and only if $p = e$, which is equivalent to the fact that $a - \lambda$ is quasi-nilpotent but not nilpotent.

Statements (i) and (ii) are well known and they can be derived, for example, from [22] and [17]. Furthermore, it is not difficult to prove that statement (i) (respectively statement (ii)) is equivalent to the existence of $0 \neq p^2 = p \in A$ such that $ap = pa$ (equivalently, $a - \lambda = p(a - \lambda)p + p'(a - \lambda)p'$, $p' = e - p$), $p(a - \lambda)p$ is nilpotent (respectively quasi-nilpotent but not nilpotent) in the Banach algebra pAp and $p'(a - \lambda)p'$ is invertible in the Banach algebra $p'Ap'$. Next consider B another unital Banach algebra and $b \in B$. Note that both the identity of B and of $\overline{A \otimes B}$ will be denoted by e .

(iii) Since $\sigma(a \otimes b) = \sigma(a)\sigma(b)$ ([12, Theorem 11.7.6]), according to [13, Theorem 6],

$$\text{iso } \sigma(a \otimes b) \setminus \{0\} = (\text{iso } \sigma(a) \setminus \{0\})(\text{iso } \sigma(b) \setminus \{0\}).$$

(iv) Set

$$\mathbb{L} = (I(a) \setminus \{0\})(I(b) \setminus \{0\}) \cup (I(a) \setminus \{0\})(\Pi(b) \setminus \{0\}) \cup (\Pi(a) \setminus \{0\})(I(b) \setminus \{0\}).$$

Then clearly, $\text{iso } \sigma(a \otimes b) \setminus \{0\} = \mathbb{L} \cup (\Pi(a) \setminus \{0\})(\Pi(b) \setminus \{0\})$.

(v) Let $\lambda \in \text{iso } \sigma(a \otimes b) \setminus \{0\}$. Then, it is not difficult to prove that there exist $n \in \mathbb{N}$ and finite sequences $\{\mu\} = \{\mu_1, \dots, \mu_n\} \subseteq \text{iso } \sigma(a) \setminus \{0\}$ and $\{\nu\} = \{\nu_1, \dots, \nu_n\} \subseteq \text{iso } \sigma(b) \setminus \{0\}$ such that $\lambda = \mu_i \nu_i$ for all $i = 1, \dots, n$, and that if there exist $\mu \in \sigma(a) \setminus \{0\}$ and $\nu \in \sigma(b) \setminus \{0\}$ such that $\lambda = \mu\nu$, then there is i_0 , $1 \leq i_0 \leq n$, for which $\mu = \mu_{i_0}$ and $\nu = \nu_{i_0}$.

In the following theorem the position of 0 in the isolated and the limit points of the $\sigma(a \otimes b)$ will be studied, where $a \in A$ and $b \in B$, A and B unital Banach algebras. To this end, the description of the spectrum and the Drazin spectrum presented in [3, Theorem 12] will be central. Note that if $0 \in \text{iso } \sigma(a \otimes b)$, then $0 \in \text{iso } \sigma(a)$ or $0 \in \text{iso } \sigma(b)$.

Theorem 3.2. *Let A and B be two unital Banach algebras and consider $a \in A$ and $b \in B$. Then, the following statements hold.*

- (i) $\sigma(a) = \Pi(a) = \{0\}$ or $\sigma(b) = \Pi(b) = \{0\}$ if and only if $\sigma(a \otimes b) = \Pi(a \otimes b) = \{0\}$.
- (ii) Suppose that neither a nor b is nilpotent. Then necessary and sufficient for $\sigma(a) = I(a) = \{0\}$ or $\sigma(b) = I(b) = \{0\}$ is that $\sigma(a \otimes b) = I(a \otimes b) = \{0\}$.
- (iii) If $0 \in \Pi(a)$ (respectively $0 \in \Pi(b)$) and b is invertible (respectively a is invertible) or if $0 \in \Pi(a) \cap \Pi(b)$, then $0 \in \Pi(a \otimes b)$.
- (iv) Suppose that a is not nilpotent (respectively b is not nilpotent). Then, if $0 \in \Pi(a)$ (respectively $0 \in \Pi(b)$) and $0 \in I(b)$ (respectively $0 \in I(a)$), then $0 \in I(a \otimes b)$.
- (v) If b (respectively a) is invertible and $0 \in I(a)$ (respectively $0 \in I(b)$), then $0 \in I(a \otimes b)$.
- (vi) If $0 \in I(a) \cap I(b)$, then $0 \in I(a \otimes b)$.
- (vii) Suppose that $\sigma(b) \neq \{0\}$ (respectively $\sigma(a) \neq \{0\}$). Then, if $0 \in \text{acc } \sigma(a)$ (respectively $0 \in \text{acc } \sigma(b)$), then $0 \in \text{acc } \sigma(a \otimes b)$.

Proof. (i). If $a \in A$ or $b \in B$ is nilpotent, then clearly $a \otimes b \in A \overline{\otimes} B$ is nilpotent. On the other hand, if $a \otimes b \in A \overline{\otimes} B$ is nilpotent, since the norm considered on $A \overline{\otimes} B$ is a crossnorm, then a or b must be nilpotent.

(ii). If $\sigma(a) = I(a) = \{0\}$ or $\sigma(b) = I(b) = \{0\}$, then $\sigma(a \otimes b) = \{0\}$. Now well, if $\sigma(a \otimes b) = \Pi(a \otimes b) = \{0\}$, then according to statement (i), a or b must be nilpotent, which is impossible. Hence, $\sigma(a \otimes b) = I(a \otimes b) = \{0\}$. To prove the converse implication, note that $\sigma(a) = \{0\}$ or $\sigma(b) = \{0\}$. However, since a and b are not nilpotent, $\sigma(a) = I(a) = \{0\}$ or $\sigma(b) = I(b) = \{0\}$.

(iii). The conditions of the statement under consideration imply that $a \in A$ and $b \in B$ are Drazin invertible. Using the Drazin inverses of $a \in A$ and $b \in B$, it is not difficult to prove that $a \otimes b \in A \overline{\otimes} B$ is Drazin invertible.

(iv) Suppose that $0 \in \Pi(a) \cap I(b)$. Clearly $0 \in \text{iso } \sigma(a \otimes b)$. According to Remark 3.1(i)-(ii), there exist $0 \neq p^2 = p \in A$ and $0 \neq q^2 = q \in B$ such that $ap = pa$, $bq = qb$, $(e - p)a(e - p)$ is invertible in $(e - p)A(e - p)$, $(e - q)b(e - q)$ is invertible in $(e - q)B(e - q)$, $pap \in pAp$ is nilpotent and $pbq \in qBq$ is quasi-nilpotent but not nilpotent. Let $r = p \otimes q + (e - p) \otimes q + p \otimes (e - q) \in A \overline{\otimes} B$. Note that $r = r^2$ and $e - r = (e - p) \otimes (e - q)$. In particular, $r \neq 0$ and $ra \otimes b = a \otimes br$. In addition, $(e - r)a \otimes b(e - r) = (e - p)a(e - p) \otimes (e - q)b(e - q)$, which is invertible in $(e - p)A(e - p) \overline{\otimes} (e - q)B(e - q) = (e - r)A \overline{\otimes} B(e - r)$. Moreover, a straightforward calculation proves that $ra \otimes br = s_1 + s_2 + s_3$, where $s_1 = pap \otimes qbq \in S_1 = pAp \overline{\otimes} qBq$, $s_2 = (e - p)a(e - p) \otimes qbq \in S_2 = (e - p)A(e - p) \overline{\otimes} qBq$ and $s_3 = pap \otimes (e - q)b(e - q) \in S_3 = pAp \overline{\otimes} (e - q)B(e - q)$. Note that since $s_i s_j = 0$, $1 \leq i \neq j \leq 3$, $(ra \otimes br)^n = s_1^n + s_2^n + s_3^n$.

Now well, since $0 \in \Pi(a) \cap I(b)$, pap is nilpotent and qbq is quasi-nilpotent but not nilpotent. In particular, there is $k \in \mathbb{N}$ such that for $n \geq k$, $(ra \otimes br)^n = s_2^n$. However, since $(e-p)a(e-p) \in (e-p)A(e-p)$ is invertible, $qbq \in qBq$ is quasi-nilpotent but not nilpotent and the norm on the tensor product is a crossnorm, $ra \otimes br$ is quasi-nilpotent but not nilpotent. Therefore, according to Remark 3.1(ii), $0 \in I(a \otimes b)$. The case $0 \in I(a) \cap \Pi(b)$ can be proved in a similar way.

(v) Adapt the argument in the proof of statement (iv) using in particular $p = 0$ (respectively $q = 0$) in the case $a \in A$ invertible (respectively $b \in B$ invertible).

(vi) Proceed as in the proof of statement (iv) but assuming that $pap \in pAp$ is quasi-nilpotent but not nilpotent. Note then that since $(ra \otimes br)^n = s_1^n + s_2^n + s_3^n$ and $s_i s_j = 0$, $1 \leq i \neq j \leq 3$, necessary and sufficient for $ra \otimes br$ to be nilpotent is that there exists $k \in \mathbb{N}$ such that $s_1^k = s_2^k = s_3^k = 0$, which, since the norm of $A \overline{\otimes} B$ is a cross norm and $pap \in pAp$ and $qbq \in qBq$ are quasi-nilpotent but not nilpotent, is impossible.

(vii) A straightforward calculation proves this statement. \square

To characterize the non-null isolated points of $\sigma(a \otimes b)$ ($a \in A$, $b \in B$, A and B unital Banach algebras), first of all a particular case need to be considered.

Theorem 3.3. *Let A and B be two unital Banach algebras and consider $a \in A$ and $b \in B$ such that $\sigma(a) = \{\mu\}$, $\sigma(b) = \{\nu\}$ and $\lambda = \mu\nu \neq 0$.*

(i) If $\sigma(a) = \Pi(a) = \{\mu\}$ and $\sigma(b) = \Pi(b) = \{\nu\}$, then $\sigma(a \otimes b) = \Pi(a \otimes b) = \{\lambda\}$.

(ii) If either $\sigma(a) = I(a) = \{\mu\}$ or $\sigma(b) = I(b) = \{\nu\}$, then $\sigma(a \otimes b) = I(a \otimes b) = \{\lambda\}$.

Proof. First of all, note that $\sigma(a \otimes b) = \{\lambda\}$.

(i). Since $a \otimes b - \lambda = (a - \mu) \otimes b + \mu \otimes (b - \nu)$, $a - \mu \in A$ and $b - \nu \in B$ are nilpotent and $(a - \mu) \otimes b$ and $\mu \otimes (b - \nu)$ commute, $a \otimes b - \lambda$ is nilpotent.

(ii) Suppose that $b - \nu \in B$ is quasi-nilpotent but not nilpotent (the case $a - \mu \in A$ is quasi-nilpotent but not nilpotent can be proved using similar arguments). Let $c = a - \mu$, $d = \mu(b - \nu)$ and note that $a \otimes b - \lambda = c \otimes b + e \otimes d$ and $c \otimes b$ and $e \otimes d$ commute. Now well, since $c \in A$ is quasi-nilpotent, according to [12, Theorem 7.4.2], there is a sequence $(h_n)_{n \in \mathbb{N}} \subseteq A$ such that $\|h_n\| = 1$ and $(ch_n)_{n \in \mathbb{N}} \subseteq A$ converges to $0 \in A$. In addition, since $d \in B$ is quasi-nilpotent but not nilpotent, for each $k \in \mathbb{N}$, $d^k \neq 0$ and $l_k = \frac{2e}{\|d^k\|}$ is such that $\|d^k l_k\| = 2$. Next consider $z_{k,j} = \frac{k!}{j!(k-j)!}$, where $1 \leq j \leq k \in \mathbb{N}$. Then, an easy calculation proves that given $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ such that

$$\left\| \left(\sum_{j=1}^k z_{j,k} c^j \otimes b^j d^{k-j} \right) h_{n_k} \otimes l_k \right\| < 1.$$

In particular,

$$\|(a \otimes b - \lambda)^k h_{n_k} \otimes l_k\| = \|(c \otimes b + e \otimes d)^k h_{n_k} \otimes l_k\| \geq \|h_{n_k} \otimes d^k l_k\| - \left\| \left(\sum_{j=1}^k z_{j,k} c^j \otimes b^j d^{k-j} \right) h_{n_k} \otimes l_k \right\| > 1.$$

Therefore, $a \otimes b - \lambda \in A \overline{\otimes} B$ is quasi-nilpotent but not nilpotent, equivalently $\lambda \in I(a \otimes b)$. \square

In the following theorem, the non-null isolated points of the spectrum of the tensor product of two Banach algebra elements will be fully characterized. To this end, note that if A is a unital Banach algebra and $p^2 = p \in A$, then the spectrum of $z \in pA$ in the unital Banach algebra pA will be denoted by $\sigma_{pA}(z)$. In addition, if $q^2 = q \in B$, then with the norm restricted from $A \overline{\otimes} B$, $pA \overline{\otimes} qB$ is a Banach algebra with a uniform crossnorm.

Theorem 3.4. *Let A and B be two unital Banach algebras and consider $a \in A$ and $b \in B$. Then, the following statements hold.*

- (i) $\mathbb{L} = I(a \otimes b) \setminus \{0\}$.
- (ii) $\Pi(a \otimes b) \setminus \{0\} = (\Pi(a) - \{0\})(\Pi(b) - \{0\}) \setminus \mathbb{L}$.

Proof. Note that according to Remark 3.1(iv), statement (i) implies statement (ii).

To prove statement (i), let $\lambda \in \text{iso } \sigma(a \otimes b) \setminus \{0\}$. Then, according to Remark 3.1(v), there exist $n \in \mathbb{N}$ and finite spectral sets $\{\mu\} = \{\mu_1, \dots, \mu_n\} \subseteq \text{iso } \sigma(a)$ and $\{\nu\} = \{\nu_1, \dots, \nu_n\} \subseteq \text{iso } \sigma(b)$ such that $\lambda = \mu_i \nu_i$, for all $1 \leq i \leq n$ and that if there exist $\mu \in \sigma(a) \setminus \{0\}$ and $\nu \in \sigma(b) \setminus \{0\}$ such that $\lambda = \mu\nu$, then there is i_0 , $1 \leq i_0 \leq n$, with the property $\mu = \mu_{i_0}$ and $\nu = \nu_{i_0}$. In addition, using the functional calculus it is not difficult to prove that there are idempotents p and $(p_i)_{i=1}^n$ in A and q and $(q_i)_{i=1}^n$ in B such that $p = \sum_{i=1}^n p_i$, $p_i p_j = 0$ and $p_i a p_j = 0$ for $i \neq j$, $p' p_i = 0$ ($p' = e - p$), $p_i a p' = 0$ and $p_i a = a p_i = p_i a p_i$, $1 \leq i \leq n$, $p a p' = p' a p = 0$, $A = pA \oplus p'A$, $pA = \bigoplus_{i=1}^n p_i A$, $\sigma_{pA}(pa) = \{\mu\}$, $\sigma_{p_i A}(p_i a) = \sigma_{pA}(p_i a) = \sigma(p_i a) = \{\mu_i\}$, $\sigma(p' a)_{p'A} = \sigma(a) \setminus \{\mu\}$ and $q = \sum_{i=1}^n q_i$, $q_i q_j = 0$ and $q_i b q_j = 0$ for $i \neq j$, $q' q_i = 0$ ($q' = e - q$), $q_i b q' = 0$ and $q_i b = b q_i = q_i b q_i$, $1 \leq i \leq n$, $B = qB \oplus q'B$, $qB = \bigoplus_{i=1}^n q_i B$, $\sigma_{qB}(qb) = \{\nu\}$, $\sigma_{q_i B}(q_i b) = \sigma_{qB}(q_i b) = \sigma(q_i b) = \{\nu_i\}$, $\sigma(q' b)_{q'B} = \sigma(b) \setminus \{\nu\}$. Next, define $r = \sum_{i=1}^n p_i \otimes q_i \in A \overline{\otimes} B$. Then, it is not difficult to prove that $0 \neq r = r^2$, $ra \otimes b = a \otimes br$ and $e - r = p \otimes q' + p' \otimes q + p' \otimes q' + \sum_{1 \leq i \neq j \leq n} p_i \otimes q_j$.

Note that $\lambda \notin \sigma_{pA \overline{\otimes} q'B}(p \otimes q' a \otimes b) = \sigma_{pA \overline{\otimes} q'B}(pa \otimes q' b) = \sigma_{pA}(pa) \sigma_{q'B}(q' b) = \{\mu\}(\sigma(b) - \{\nu\})$. However, a straightforward calculation, using in particular this latter fact, proves that $z_1 = p \otimes q' a \otimes b p \otimes q' - \lambda p \otimes q'$ is invertible in $p \otimes q' A \overline{\otimes} B p \otimes q'$. What is more, similar arguments prove that $z_2 = p' \otimes q a \otimes b p' \otimes q - \lambda p' \otimes q$ is invertible in $p' \otimes q A \overline{\otimes} B p' \otimes q$, $z_3 = p' \otimes q' a \otimes b p' \otimes q' - \lambda p' \otimes q'$ is invertible in $p' \otimes q' A \overline{\otimes} B p' \otimes q'$ and $z_{i,j} = p_i \otimes q_j a \otimes b p_i \otimes q_j - \lambda p_i \otimes q_j$ is invertible in $p_i \otimes q_j A \overline{\otimes} B p_i \otimes q_j$, for $1 \leq i \neq j \leq n$. Now well, according to the properties of the idempotents p , p_i , q and q_i , $1 \leq i \leq n$, recalled in the previous paragraph, it is not difficult to prove that $(e - r)(a \otimes b - \lambda)(e - r) = z_1 + z_2 + z_3 + \sum_{1 \leq i \neq j \leq n} z_{i,j}$, $(e - r)A \overline{\otimes} B(e - r) = pA p' \overline{\otimes} q' B q' \oplus p' A p' \overline{\otimes} q B q \oplus p' A p' \overline{\otimes} q' B q' \oplus \sum_{1 \leq i \neq j \leq n} p_i A p_i \overline{\otimes} q_j B q_j$ and $(e - r)(a \otimes b - \lambda)(e - r)$ is invertible in $(e - r)A \overline{\otimes} B(e - r)$.

Moreover, using again the properties of the aforementioned idempotents, it is possible to prove that $rA \overline{\otimes} B r = \bigoplus_{1 \leq i \leq n} p_i A p_i \overline{\otimes} q_i B q_i$ and $r(a \otimes b - \lambda)r = \sum_{1 \leq i \leq n} p_i \otimes q_i (a \otimes b - \lambda) p_i \otimes q_i$. What is more, since $p_i \otimes q_i (a \otimes b - \lambda) p_i \otimes q_i$ is quasi-nilpotent in $p_i \otimes q_i A \overline{\otimes} B p_i \otimes q_i$ ($1 \leq i \leq n$), $r(a \otimes b - \lambda)r$ is quasi-nilpotent. Note that $r(a \otimes b - \lambda)r$ is nilpotent if and only if $p_i \otimes q_i (a \otimes b - \lambda) p_i \otimes q_i$ is nilpotent for all $i = 1, \dots, n$.

Let $\lambda \in I(a \otimes b) \setminus \{0\}$. Suppose that for all $i = 1, \dots, n$, $\mu_i \in \Pi(a)$ and $\nu_i \in \Pi(b)$. In particular, according to Remark 3.1(i) and the properties of the idempotents p_i and q_i , $1 \leq i \leq n$, $\sigma_{p_i A}(p_i a) = \Pi(p_i a) = \{\mu_i\}$ and $\sigma_{q_i B}(q_i b) = \Pi(q_i b) = \{\nu_i\}$. However, according to Theorem 3.3(i), $\sigma_{p_i A \overline{\otimes} q_i B}(p_i a \otimes q_i b) = \Pi(p_i a \otimes q_i b) = \{\mu_i \nu_i\} = \{\lambda\}$ ($1 \leq i \leq n$). As a result, $p_i a p_i \otimes q_i b q_i - \lambda p_i \otimes q_i = p_i a \otimes q_i b - \lambda p_i \otimes q_i$ is nilpotent in $p_i A \overline{\otimes} q_i B$, and hence in $p_i A p_i \overline{\otimes} q_i B q_i$ ($i = 1, \dots, n$). Consequently, $r(a \otimes b - \lambda)r \in rA \overline{\otimes} B r$ is nilpotent. Now well, according to what has been proved, $(e - r)(a \otimes b - \lambda)(e - r)$ is invertible in $(e - r)A \overline{\otimes} B(e - r)$. Then, according to Remark 3.1(i) and its equivalent formulation (see Remark 3.1), $\lambda \in \Pi(a \otimes b)$, which is impossible. Therefore, $I(a \otimes b) \setminus \{0\} \subseteq \mathbb{L}$.

On the other hand, if $\lambda \in \mathbb{L}$, then according to Remark 3.1(ii) and the properties of the idempotents p_i and q_i , $1 \leq i \leq n$, there is i_0 , $1 \leq i_0 \leq n$, such that $\sigma_{p_{i_0} A}(p_{i_0} a) = I(p_{i_0} a) = \{\mu_{i_0}\}$ or $\sigma_{q_{i_0} B}(q_{i_0} b) = I(q_{i_0} b) = \{\nu_{i_0}\}$. Therefore, according to Theorem 3.3(ii), $\sigma_{p_{i_0} A \overline{\otimes} q_{i_0} B}(p_{i_0} a \otimes q_{i_0} b) = I(p_{i_0} a \otimes q_{i_0} b) = \{\mu_{i_0} \nu_{i_0}\} = \{\lambda\}$. Then, $p_{i_0} a p_{i_0} \otimes q_{i_0} b q_{i_0} - \lambda p_{i_0} \otimes q_{i_0} = p_{i_0} a \otimes q_{i_0} b - \lambda p_{i_0} \otimes q_{i_0}$ is quasi-nilpotent but not nilpotent in $p_{i_0} A \overline{\otimes} q_{i_0} B$, and hence in $p_{i_0} A p_{i_0} \overline{\otimes} q_{i_0} B q_{i_0}$. As a result, $r(a \otimes b - \lambda)r \in rA \overline{\otimes} B r$ is quasi-nilpotent but not nilpotent. Now well, as in the previous paragraph, $(e - r)(a \otimes b - \lambda)(e - r)$ is invertible in $(e - r)A \overline{\otimes} B(e - r)$. Then, according to

Remark 3.1(ii) and its equivalent formulation (see Remark 3.1), $\lambda \in I(a \otimes b) \setminus \{0\}$. Thus, $\mathbb{L} \subseteq I(a \otimes b) \setminus \{0\}$. \square

4. The Drazin spectrum

In this section, the Drazin spectrum of the tensor product of two Banach algebra elements will be characterized. To this end, given A and B two unital Banach algebras and $a \in A$ and $b \in B$, consider the sets $\mathbb{A} = \sigma(a)(\text{acc } \sigma(b)) \cup (\text{acc } \sigma(a))\sigma(b)$ and $\mathbb{B} = I(a)I(b) \cup I(a)\Pi(b) \cup \Pi(a)I(b)$. Note that according to [13, Theorem 6] and Theorem 3.4, $\mathbb{A} \setminus \{0\} = \text{acc } \sigma(a \otimes b) \setminus \{0\}$ and $\mathbb{B} \setminus \{0\} = \mathbb{L} = I(a \otimes b) \setminus \{0\}$, respectively. Recall in addition that, according to [3, Theorem 12(i)-(ii)], given a unital Banach algebra A and $a \in A$, necessary and sufficient for $\sigma_{DR}(a) = \emptyset$ is that $\sigma(a) = \Pi(a)$. In the following theorem, the relationship between $\sigma_{DR}(a \otimes b)$ and $\mathbb{D} = \sigma(a)\sigma_{DR}(b) \cup \sigma_{DR}(a)\sigma(b)$ will be studied.

Theorem 4.1. *Let A and B be two unital Banach algebras and consider $a \in A$ and $b \in B$. Then, the following statements hold.*

- (i) *If $\sigma(a) = \Pi(a)$ and $\sigma(b) = \Pi(b)$, then $\mathbb{D} = \emptyset = \sigma_{DR}(a \otimes b)$.*
- (ii) *If $\sigma_{DR}(a) \neq \emptyset$ or $\sigma_{DR}(b) \neq \emptyset$, then $\sigma_{DR}(a \otimes b) \setminus \{0\} = \mathbb{D} \setminus \{0\}$ and $\sigma_{DR}(a \otimes b) \subseteq \mathbb{D}$.*

Proof. (i) Since the condition in the statement under consideration is equivalent to $\sigma_{DR}(a) = \emptyset$ and $\sigma_{DR}(b) = \emptyset$, $\mathbb{D} = \emptyset$. On the other hand, according to Theorem 3.4(i) and Theorem 3.2(iii), $\sigma(a \otimes b) = \Pi(a)\Pi(b) = \text{iso } \sigma(a \otimes b) = \Pi(a \otimes b)$, which implies that $\sigma_{DR}(a \otimes b) = \emptyset$.

(ii) Suppose that $\sigma(a) = \pi(a)$ and $\sigma_{DR}(b) \neq \emptyset$. Then, $\mathbb{D} \setminus \{0\} = (\Pi(a) \setminus \{0\})(\text{acc } \sigma(b) \setminus \{0\}) \cup (\Pi(a) \setminus \{0\})(I(b) \setminus \{0\}) = (\mathbb{A} \setminus \{0\}) \cup (\mathbb{B} \setminus \{0\}) = \sigma_{DR}(a \otimes b) \setminus \{0\}$.

Note that if $0 \notin \sigma(a \otimes b)$, then $\sigma_{DR}(a \otimes b) = \mathbb{D}$. As a result, to prove that $\sigma_{DR}(a \otimes b) \subseteq \mathbb{D}$, it is enough to prove that if $0 \in \sigma(a \otimes b) \setminus \mathbb{D}$, then $0 \notin \sigma_{DR}(a \otimes b)$. Now well, since $\sigma(a \otimes b) = \Pi(a)\Pi(b) \cup \mathbb{D}$, if $0 \in \sigma(a \otimes b) \setminus \mathbb{D}$, then it is not difficult to prove that $0 \notin \Pi(a)$ and $0 \in \Pi(b)$. However, according to Theorem 3.2(iii), $0 \in \Pi(a \otimes b)$.

The case $\sigma_{DR}(a) \neq \emptyset$ and $\sigma(b) = \Pi(b)$ can be proved interchanging a with b .

Next suppose that $\sigma_{DR}(a) \neq \emptyset$ and $\sigma_{DR}(b) \neq \emptyset$. Using arguments similar to the ones considered before, it is easy to prove that $\mathbb{D} \setminus \{0\} = (\mathbb{A} \setminus \{0\}) \cup (\mathbb{B} \setminus \{0\}) = \sigma_{DR}(a \otimes b) \setminus \{0\}$ ([3, Theorem 12(iv)]). In addition, as before, if $0 \notin \sigma(a \otimes b)$, then $\sigma_{DR}(a \otimes b) = \mathbb{D}$. On the other hand, if $0 \in \sigma(a \otimes b)$, then $0 \in \sigma(a)$ or $0 \in \sigma(b)$. However, since $\sigma_{DR}(a) \neq \emptyset$ and $\sigma_{DR}(b) \neq \emptyset$, $0 \in \mathbb{D}$. Therefore, $\sigma_{DR}(a \otimes b) \subseteq \mathbb{D}$. \square

Under the same conditions of Theorem 4.1, note that $\sigma_{DR}(a \otimes b)$ and \mathbb{D} , in general, do not coincide. In fact, if $a \in A$ is nilpotent (equivalently if $\sigma(a) = \Pi(a) = \{0\}$) and $b \in B$ is such that $\sigma_{DR}(b) \neq \emptyset$, then $\sigma_{DR}(a \otimes b) = \emptyset$ and $\mathbb{D} = \{0\}$. In fact, in this case $a \otimes b \in A \otimes B$ is nilpotent, $\sigma(a \otimes b) = \Pi(a \otimes b) = \{0\}$, $\sigma_{DR}(a \otimes b) = \emptyset$ and $\mathbb{D} = \{0\}$. Note, however, that if $a \in A$ is nilpotent and $\sigma_{DR}(b) = \emptyset$, then according to Theorem 4.1(i), $\sigma_{DR}(a \otimes b) = \emptyset = \mathbb{D}$. In addition, according to Theorem 4.1(ii), $\sigma_{DR}(a \otimes b) \subsetneq \mathbb{D}$ if and only if $\mathbb{D} = \sigma_{DR}(a \otimes b) \cup \{0\}$, $0 \notin \sigma_{DR}(a \otimes b)$, which in turn is equivalent to the fact that $0 \in \mathbb{D} \cap \Pi(a \otimes b)$. In the following theorems it will be characterized when $\sigma_{DR}(a \otimes b)$ and \mathbb{D} coincide. In first place, the case in which only one Drazin spectrum is the empty set will be studied. Naturally, nilpotent elements will not be considered.

Theorem 4.2. *Let A and B be two unital Banach algebras and consider $a \in A$ and $b \in B$.*

- (i) *If $\sigma(a) = \Pi(a) \neq \{0\}$ and $\sigma_{DR}(b) \neq \emptyset$, then necessary and sufficient for $\sigma_{DR}(a \otimes b)$ and \mathbb{D} to coincide is that $0 \notin \Pi(a)$ or $0 \notin \rho_{DR}(b)$.*
- (ii) *If $\sigma_{DR}(a) \neq \emptyset$ and $\sigma(b) = \Pi(b) \neq \{0\}$, then necessary and sufficient for $\sigma_{DR}(a \otimes b)$ and \mathbb{D} to coincide is that $0 \notin \rho_{DR}(a)$ or $0 \notin \Pi(b)$.*

Proof. (i) Note that $\mathbb{D} = \Pi(a)\sigma_{DR}(b)$. Suppose that $\sigma_{DR}(a \otimes b) = \mathbb{D}$. Now well, if $0 \in \Pi(a)$ and $0 \in \rho_{DR}(b)$, then it is clear that $0 \in \mathbb{D}$ and, according to Theorem 3.2(iii), $0 \in \Pi(a \otimes b)$. In particular, $0 \in \mathbb{D} \setminus \sigma_{DR}(a \otimes b)$, which is impossible.

To prove the converse, suppose that a is invertible or b is not Drazin invertible. If $\sigma_{DR}(a \otimes b)$ and \mathbb{D} does not coincide, then $0 \in \mathbb{D} \setminus \sigma_{DR}(a \otimes b)$ thanks to Theorem 4.1(ii). Now well, if $0 \notin \sigma(a) = \Pi(a)$, then $0 \in \sigma_{DR}(b) = \text{acc } \sigma(b) \cup I(b)$. Clearly, if $0 \in \text{acc } \sigma(b)$, then $0 \in \text{acc } \sigma(a \otimes b) \subseteq \sigma_{DR}(a \otimes b)$, while if $0 \in I(b)$, then according to Theorem 3.2(v), $0 \in I(a \otimes b) \subseteq \sigma_{DR}(a \otimes b)$, which is impossible. On the other hand, suppose that $0 \in \Pi(a)$ and $0 \in \sigma_{DR}(b) = \text{acc } \sigma(b) \cup I(b)$. If $0 \in \text{acc } \sigma(b)$, then since $\sigma(a) = \Pi(a) \neq \{0\}$, $0 \in \text{acc } \sigma(a \otimes b) \subseteq \sigma_{DR}(a \otimes b)$, while if $0 \in I(b)$, then according to Theorem 3.2(iv), $0 \in I(a \otimes b) \subseteq \sigma_{DR}(a \otimes b)$, which is impossible.

(ii) Interchange a with b and use the same argument. \square

Note that to prove the following theorem, the description of the spectrum and the Drazin spectrum considered in [3, Theorem 12] will be central.

Theorem 4.3. *Let A and B be two unital Banach algebras and consider $a \in A$ and $b \in B$ such $\sigma_{DR}(a) \neq \emptyset$ and $\sigma_{DR}(b) \neq \emptyset$. Then, the following statements are equivalent.*

- (i) $\mathbb{D} = \sigma_{DR}(a \otimes b)$;
- (ii) $a \in A$ and $b \in B$ are invertible or $a \otimes b$ is not Drazin invertible;
- (iii) $a \in A$ and $b \in B$ are invertible or $0 \notin (\Pi(a) \cap \rho_{DR}(b) \cup \rho_{DR}(a) \cap \Pi(b))$.

Proof. (i) \Rightarrow (ii). If $a \in A$ or $b \in B$ is not invertible, then $0 \in \sigma(a \otimes b)$. Now well, since $\sigma_{DR}(a) \neq \emptyset$ and $\sigma_{DR}(b) \neq \emptyset$, $0 \in \mathbb{D} = \sigma_{DR}(a \otimes b)$. In particular, $a \otimes b$ is not Drazin invertible.

(ii) \Rightarrow (iii). Suppose that $0 \in \sigma(a \otimes b)$. If $0 \in \Pi(a) \cap \rho_{DR}(b)$ or $0 \in \rho_{DR}(a) \cap \Pi(b)$, then according to Theorem 3.2(iii), $0 \in \Pi(a \otimes b)$, which is impossible.

(iii) \Rightarrow (i). If $a \in A$ and $b \in B$ are invertible, then $0 \notin \sigma(a \otimes b)$ and, according to Theorem 4.1(ii), $\mathbb{D} = \sigma_{DR}(a \otimes b)$. On the other hand, if $0 \in \sigma(a \otimes b)$, then $0 \in \sigma(a)$ or $0 \in \sigma(b)$, and as before, $0 \in \mathbb{D}$. Suppose that $0 \in \sigma(a)$ (the case $0 \in \sigma(b)$ can be proved interchanging a with b). To prove that $0 \in \sigma_{DR}(a \otimes b)$, several cases must be considered.

If $0 \in \Pi(a)$, then since a is not nilpotent ($\sigma_{DR}(a) \neq \emptyset$), there is $\mu \in \sigma(a) \setminus \{0\}$. In particular, if $0 \in \text{acc } \sigma(b)$, then $0 \in \text{acc } \sigma_{DR}(a \otimes b) \subseteq \sigma_{DR}(a \otimes b)$. In addition, if $0 \in I(b)$, according to Theorem 3.2(iv), $0 \in I(a \otimes b) \subseteq \sigma_{DR}(a \otimes b)$.

Next suppose that $0 \in \text{acc } \sigma(a)$. If b is invertible or if $0 \in \text{acc } \sigma(b)$, then clearly $0 \in \text{acc } \sigma_{DR}(a \otimes b) \subseteq \sigma_{DR}(a \otimes b)$. If $0 \in \Pi(b)$, then as before, since b is not nilpotent, there exists $\nu \in \sigma(b) \setminus \{0\}$. In particular, $0 \in \text{acc } \sigma_{DR}(a \otimes b) \subseteq \sigma_{DR}(a \otimes b)$. If $0 \in I(b)$, then there are two possibilities. If there is $\nu \in \sigma(b) \setminus \{0\}$, then $0 \in \text{acc } \sigma_{DR}(a \otimes b) \subseteq \sigma_{DR}(a \otimes b)$, while if $\sigma(b) = I(b) = \{0\}$, according to Theorem 3.2(ii), $0 \in I(a \otimes b) \subseteq \sigma_{DR}(a \otimes b)$.

Now consider the case $0 \in I(a)$. As before, if $\sigma(a) = I(a) = \{0\}$, then since b is not nilpotent, $0 \in I(a \otimes b) \subseteq \sigma_{DR}(a \otimes b)$. Concerning the case $\sigma(a) \setminus \{0\} \neq \emptyset$, if $0 \in (\rho(b) \cup \Pi(b) \cup I(b))$, then according to Theorem 3.2(iv-vi), $0 \in I(a \otimes b) \subseteq \sigma_{DR}(a \otimes b)$, while if $0 \in \text{acc } \sigma(b)$, $0 \in \text{acc } \sigma(a \otimes b) \subseteq \sigma_{DR}(a \otimes b)$. \square

5. Elementary operators

Recall that given Banach spaces X and Y and Banach space operator $S \in L(X)$ and $T \in L(Y)$, the elementary operator defined by S and T is $\tau_{ST}: L(Y, X) \rightarrow L(Y, X)$, $\tau_{ST}(A) = SAT$, where $A \in L(Y, X)$ and $L(Y, X)$ stands for the Banach space of all bounded linear maps from Y into X . In this section the Drazin spectrum of elementary operators between Banach spaces will be studied. To this end, however, some preparation is needed. In first place, the definition of the axiomatic tensor product of Banach spaces introduced in [11] will be recalled.

A pair $\langle X, \tilde{X} \rangle$ of Banach spaces will be called a dual pairing, if

$$(A) \tilde{X} = X^* \text{ or } (B) X = \tilde{X}^*,$$

where X^* denotes the dual space of X . In both cases, the canonical bilinear mapping is denoted by

$$X \times \tilde{X} \rightarrow \mathbb{C}, (x, u) \mapsto \langle x, u \rangle.$$

Given $\langle X, \tilde{X} \rangle$ a dual pairing, consider the subalgebra $\mathcal{L}(X)$ of $L(X)$ consisting of all operators $T \in L(X)$ for which there is an operator $T' \in L(\tilde{X})$ with

$$\langle Tx, u \rangle = \langle x, T'u \rangle,$$

for all $x \in X$ and $u \in \tilde{X}$. It is clear that if the dual pairing is $\langle X, X^* \rangle$, then $\mathcal{L}(X) = L(X)$, and that if the dual pairing is $\langle X^*, X \rangle$, then $\mathcal{L}(X^*) = \{T^* \in L(X^*) : T \in L(X)\}$, where $T^* \in L(X^*)$ denotes the adjoint map of $T \in L(X)$. In particular, each operator of the form

$$f_{y,v} : X \rightarrow X, x \mapsto \langle x, v \rangle y,$$

is contained in $\mathcal{L}(X)$, for $y \in X$ and $v \in \tilde{X}$.

Next the definition of the axiomatic tensor product given in [11] will be recalled.

Definition 5.1. *Given two dual pairings $\langle X, \tilde{X} \rangle$ and $\langle Y, \tilde{Y} \rangle$, a tensor product of the Banach spaces X and Y relative to the dual pairings $\langle X, \tilde{X} \rangle$ and $\langle Y, \tilde{Y} \rangle$ is a Banach space Z together with two continuous bilinear mappings*

$$X \times Y \rightarrow Z, (x, y) \mapsto x \otimes y; \quad \mathcal{L}(X) \times \mathcal{L}(Y) \mapsto L(Z), \quad (T, S) \mapsto T \otimes S,$$

which satisfy the following conditions,

- (T1) $\|x \otimes y\| = \|x\| \|y\|$,
- (T2) $T \otimes S(x \otimes y) = (Tx) \otimes (Sy)$,
- (T3) $(T_1 \otimes S_1) \circ (T_2 \otimes S_2) = (T_1 T_2) \otimes (S_1 S_2), I \otimes I = I$,
- (T4) $R(f_{x,u} \otimes I) \subseteq \{x \otimes y : y \in Y\}, R(f_{y,v} \otimes I) \subseteq \{x \otimes y : x \in X\}$.

As in [11], $X \tilde{\otimes} Y$ will be written instead of Z . In addition, in [11] two main applications of Definition 5.1 were considered, namely, the completion $X \tilde{\otimes}_\alpha Y$ of the algebraic tensor product of the Banach spaces X and Y with respect to a quasi-uniform crossnorm α , see [14], and an operator ideal between Banach spaces, which will be the case considered in this section.

Definition 5.2. *An operator ideal J between Banach spaces Y and X is a linear subspace of $L(Y, X)$ equipped with a space norm α such that*

- i) $x \otimes y' \in J$ and $\alpha(x \otimes y') = \|x\| \|y'\|$,
 - ii) $SAT \in J$ and $\alpha(SAT) \leq \|S\| \alpha(A) \|T\|$,
- for $x \in X, y' \in Y', A \in J, S \in L(X), T \in L(Y)$, and $x \otimes y'$ is the usual rank one operator $Y \rightarrow X, y \mapsto \langle y, y' \rangle x$.

Examples of this kind of ideals were given in [11, 21].

Recall that, under the conditions of Definition 5.2, an operator ideal J is naturally a tensor product relative to $\langle X, X' \rangle$ and $\langle Y', Y \rangle$, with the bilinear mappings

$$X \times Y' \rightarrow J, (x, y') \mapsto x \otimes y',$$

$$\mathcal{L}(X) \times \mathcal{L}(Y') \rightarrow L(J), (S, T') \mapsto S \otimes T',$$

where $S \otimes T'(A) = SAT$. In particular, when $J = L(L(Y, X))$, $S \otimes T = \tau_{ST}$, the elementary operator defined by S and T .

Recall, in addition, that according to [11, Corollaries 3.3-3.4], $\sigma(\tau_{ST}) = \sigma(S)\sigma(T)$. Furthermore, note that if Z is a tensor product relative to $\langle X, X^* \rangle$ and $\langle Y^*, Y \rangle$, and if $M \subseteq X$ and $N \subseteq Y$ are closed and complemented subspaces of X and Y respectively, then $P \otimes Q^*(Z) \subseteq Z$ is a tensor product relative to $\langle M, M^* \rangle$ and $\langle N^*, N \rangle$, where $P^2 = P \in L(X)$ and $Q^2 = Q \in L(Y)$ are such that $R(P) = M$ and $R(Q) = N$ (actually, similar results can be proved for all kinds of tensor product satisfying Definition 5.1, however, since the main objective of this section concerns operators ideals, this case has been focused on). On the other hand, if W is a Banach space and $V \in L(W)$, then according to [20, p. 139] and [3, Theorems 3 and 12], $\Pi(V^*) = \Pi(V)$ and $\sigma_{DR}(V^*) = \sigma_{DR}(V)$, which implies that $I(V^*) = I(V)$ (naturally $\sigma(V^*) = \sigma(V)$, $\text{acc } \sigma(V^*) = \text{acc } \sigma(V)$ and $\text{iso } \sigma(V^*) = \text{iso } \sigma(V)$).

Now well, it is not difficult to check that arguments similars to the ones in the proofs of Theorems 3.2-3.4 can be used, considering in particular what has been recalled in the previous paragraph, to characterize the isolated points of $\sigma(S \otimes T)$, $S \otimes T \in L(Z)$, Z a tensor product relative to dual pairings $\langle X, \tilde{X} \rangle$ and $\langle Y, \tilde{Y} \rangle$, X and Y Banach spaces and $S \in L(X)$, $T \in L(Y)$. Therefore, the isolated points of $\sigma(\tau_{ST})$ can be fully described as it has been done in section 2, where $\tau_{ST} \in L(J)$, J an operator ideal between Banach spaces Y and X that satisfies Definition 5.2, $S \in L(X)$ and $T \in L(Y)$; this result naturally applies to $J = L(L(Y, X))$. Furthermore, the arguments in Theorems 4.1-4.3 can be adapted to the objects satisfying Definitions 5.1-5.2. As a result, the Drazin spectrum of elementary operators is fully characterized. Note that in the following theorem, $\mathbb{D} = \sigma(S)\sigma_{DR}(T) \cup \sigma_{DR}(S)\sigma(T)$, S and T as before.

Theorem 5.3. *Let X and Y be two Banach spaces and consider $S \in L(X)$ and $T \in L(Y)$. Then, the following statements hold.*

- (i) *If $\sigma(S) = \Pi(S)$ and $\sigma(T) = \Pi(T)$, then $\mathbb{D} = \emptyset = \sigma_{DR}(\tau_{ST})$.*
- (ii) *If $\sigma_{DR}(S) \neq \emptyset$ or $\sigma_{DR}(T) \neq \emptyset$, then $\sigma_{DR}(\tau_{ST}) \setminus \{0\} = \mathbb{D} \setminus \{0\}$, $\sigma_{DR}(\tau_{ST}) \subseteq \mathbb{D}$, and if $\sigma_{DR}(\tau_{ST}) \subsetneq \mathbb{D}$, then $\mathbb{D} = \sigma_{DR}(\tau_{ST}) \cup \{0\}$, $0 \notin \sigma_{DR}(\tau_{ST})$.*
- (iii) *If $\sigma(S) = \Pi(S) \neq \{0\}$ and $\sigma_{DR}(T) \neq \emptyset$, then necessary and sufficient for $\sigma_{DR}(\tau_{ST})$ and \mathbb{D} to coincide is that $0 \notin \Pi(S)$ or $0 \notin \rho_{DR}(T)$.*
- (iv) *If $\sigma_{DR}(S) \neq \emptyset$ and $\sigma(T) = \Pi(T) \neq \{0\}$, then necessary and sufficient for $\sigma_{DR}(\tau_{ST})$ and \mathbb{D} to coincide is that $0 \notin \rho_{DR}(S)$ or $0 \notin \Pi(T)$.*
- (v) *Suppose that $\sigma_{DR}(S) \neq \emptyset$ and $\sigma_{DR}(T) \neq \emptyset$. Then, the following statements are equivalent.*
- (va) $\mathbb{D} = \sigma_{DR}(\tau_{ST})$;
- (vb) S and T are invertible or τ_{ST} is not Drazin invertible;
- (vc) S and T are invertible or $0 \notin (\Pi(S) \cap \rho_{DR}(T) \cup \rho_{DR}(S) \cap \Pi(T))$.

Proof. Adapt the proofs of Theorems 4.1-4.3 to the case under consideration. Details are left to the reader. \square

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